

Particle Dynamics And Emergent Gravity

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Abstract

The emergent gravity proposal is examined within the framework of noncommutative QED/gravity correspondence from particle dynamics point of view.

1 Introduction

There have been arguments supporting the idea that the quantum theory of gravity, as the theory governing the quantum fluctuations of spacetime, might be formulated most naturally on noncommutative (NC) spacetime [1, 2]. One of the proposals in this context is that, maybe gravity should be considered as an emergent phenomena rather than a fundamental one, the so-called “emergent gravity” scenario [3, 4, 5, 6]. Accordingly, it might be likely that the gravity effects are simply an interpretation of the interaction of NC gauge fields and matter. In particular, it is observed that, at least in some available expansion in NC parameter, a proper rearrangement of interaction terms of a NC U(1) background with matter field can be interpreted as a free theory but in curved background [3, 4, 5, 6]. In [6] the idea is pushed forward from the classical level to quantum one, leading to the novel observation that the so-called UV/IR mixing phenomena [7] plays an essential role once one wants to make correspondence between the one-loop effective action in gauge theory and gravity sides. In another observation of this kind it is seen that the relative dynamics of two massive NC photons at low energy is described by a free theory but in a modified metric [8]. This sounds it is highly expected that the gravity effects for both the ordinary matter as well as the massless NC photons, which eventually come as the dynamical degrees of freedom of the resulting theory of gravity, are governed by the same NC U(1) theory.

In this note the aim is to present working examples by which the basic features and the extent of the proposed NC U(1)/gravity correspondence can be understood from the particle dynamics point of view. In particular it is shown that how the NC U(1) backgrounds that satisfy certain condition can turn to a curved background, though in some special gauges of diffeomorphism transformations. The NC U(1)/gravity correspondence is examined by the classical equation of motion of the particle in the final section.

Here we consider the canonical NC spacetime, whose coordinates satisfy the algebra

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu} \mathbf{1}, \quad (1)$$

in which $\theta^{\mu\nu}$ is an antisymmetric constant tensor and $\mathbf{1}$ represents the unit operator. It has been understood that the longitudinal directions of D-branes in the presence of a constant B-field background appear to be noncommutative, as seen by the ends of open strings [9]. It is understood that theories on canonical NC spacetime are defined by actions that are essentially the same as in ordinary spacetime, with the exception that the products between fields are replaced by \star -products, defined for two functions f and g by

$$(f \star g)(x) = \exp\left(\frac{i\theta^{\mu\nu}}{2} \partial_{x_\mu} \partial_{y_\nu}\right) f(x) g(y) \Big|_{y=x}. \quad (2)$$

2 U(1) Background As Curved Background

The starting point is the action

$$S = \int d^D x dt \left(\frac{1}{2m} \left(\hbar \nabla \psi - i [\mathbf{A}, \psi]_\star \right) \cdot \left(\hbar \nabla \psi^* - i [\mathbf{A}, \psi^*]_\star \right) - i \psi^* \left(\hbar \partial_t \psi - i [A_0, \psi]_\star \right) \right) \quad (3)$$

in which $[a, b]_\star := a \star b - b \star a$, and ψ^* is the complex conjugate of ψ . The equation of motion by the above action presents the quantum mechanics of a particle in presence of the NC U(1) background (A_0, \mathbf{A}) in the Schrödinger picture. We mention that here the matter ψ interacts in the sense of adjoint representation with the background. This kind of interaction is absent in U(1) theory on ordinary spacetime, as there the product is commutative, opposed to \star -product here. In the following we set $\hbar = 1$ for sake of simplicity; whenever needed \hbar 's can be restored by dimensional considerations.

The action above is invariant under the gauge transformations:

$$\begin{aligned}\psi &\rightarrow \psi' = U \star \psi \star U^* \\ \mathbf{A} &\rightarrow \mathbf{A}' = U \star \mathbf{A} \star U^* + i U \star \nabla U^* \\ A_0 &\rightarrow A'_0 = U \star A_0 \star U^* + i U \star \partial_t U^*\end{aligned}\tag{4}$$

in which U is a \star -phase defined by

$$U = \exp_\star(i\Lambda) = 1 + i\Lambda - \frac{1}{2}\Lambda \star \Lambda + \dots\tag{5}$$

with Λ as an arbitrary function. One easily can show that $U \star U^* = U^* \star U = 1$.

Here we consider the scattering of the particle by the given background at the lowest level. In the following we assume that 1) $A_0 = 0$, and 2) the noncommutativity is restricted to spatial directions, $\theta^{0i} = 0$. We take that the incoming and outgoing particles have momenta \mathbf{p}_1 and \mathbf{p}_2 , respectively. The expression for the transition amplitude can simply be obtained via the Feynman rules of the field theory set up of the problem

$$T_{1 \rightarrow 2} = \frac{i}{m} \sin\left(\frac{\mathbf{p}_1 \times \mathbf{p}_2}{2}\right) (\mathbf{p}_1 + \mathbf{p}_2) \cdot \tilde{\mathbf{A}}(\mathbf{q}, \omega)\tag{6}$$

in which $\mathbf{k} \times \mathbf{l} = \theta^{ij} k_i l_j$, and $\tilde{\mathbf{A}}(\mathbf{q}, \omega)$ is the Fourier transform of the background $\mathbf{A}(\mathbf{x}, t)$,

$$\tilde{\mathbf{A}}(\mathbf{q}, \omega) = \int d^D x \, dt \, e^{i\mathbf{q} \cdot \mathbf{x} - i\omega t} \mathbf{A}(\mathbf{x}, t)\tag{7}$$

with $\mathbf{q} = \mathbf{p}_2 - \mathbf{p}_1$ and $\omega = E_2 - E_1$, as transferred momenta and energy, respectively. We mention a certain combination of the incoming and outgoing particles's momenta determine the strength of the interaction. In fact in higher orders higher powers of momenta come in the expression. So, as we are working at the lowest order, we take the sine equal to its argument, getting

$$T_{1 \rightarrow 2} = \frac{i}{2m} (\mathbf{p}_1 \times \mathbf{p}_2) (\mathbf{p}_1 + \mathbf{p}_2) \cdot \tilde{\mathbf{A}}(\mathbf{q}, \omega).\tag{8}$$

Now in quantum mechanical interpretation of the problem on ordinary spacetime, the result above corresponds to the first order Born approximation expression

$$T_{1 \rightarrow 2} = \int dt \, e^{-i(E_2 - E_1)t} \langle \mathbf{p}_1 | \hat{V}(\hat{\mathbf{x}}, \hat{\mathbf{p}}, t) | \mathbf{p}_2 \rangle\tag{9}$$

with

$$\hat{H} = \frac{\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}}{2m} + \hat{V}(\hat{\mathbf{x}}, \hat{\mathbf{p}}, t), \quad \langle \mathbf{x} | \mathbf{p} \rangle = e^{i\mathbf{p} \cdot \mathbf{x}}$$

$$[\hat{x}^i, \hat{x}^j] = [\hat{p}_i, \hat{p}_j] = 0, \quad [\hat{x}^i, \hat{p}_j] = i\hbar \delta_j^i \mathbf{1}. \quad (10)$$

By comparing (8) and (9), recalling the Hermiticity of Hamiltonian, and with $\hat{\mathbf{A}} = \hat{\mathbf{A}}(\hat{\mathbf{x}}, t)$, one has the following expression for \hat{V}

$$\hat{V}(\hat{\mathbf{x}}, \hat{\mathbf{p}}, t) = \frac{i}{4m} \theta^{ij} \left\{ \hat{p}_i (\hat{\mathbf{A}} \cdot \hat{\mathbf{p}} + \hat{\mathbf{p}} \cdot \hat{\mathbf{A}}) \hat{p}_j - \hat{p}_j (\hat{\mathbf{A}} \cdot \hat{\mathbf{p}} + \hat{\mathbf{p}} \cdot \hat{\mathbf{A}}) \hat{p}_i \right\} \quad (11)$$

Using the identities $\hat{p}_i \hat{A}_j = \hat{A}_j \hat{p}_i - i \partial_i \hat{A}_j$, and $\theta^{ij} \hat{p}_i \hat{p}_j = 0$, one finds

$$\hat{V}(\hat{\mathbf{x}}, \hat{\mathbf{p}}, t) = \frac{1}{2m} (\hat{\gamma}^{ij} \hat{p}_i \hat{p}_j - i \partial_i \hat{\gamma}^{ij} \hat{p}_j) \quad (12)$$

in which

$$\hat{\gamma}^{ij}(\hat{\mathbf{x}}, t) = \theta^{kj} \partial_k \hat{A}^i + \theta^{ki} \partial_k \hat{A}^j = \hat{\gamma}^{ji}(\hat{\mathbf{x}}, t) \quad (13)$$

satisfying

$$\partial_i \partial_j \hat{\gamma}^{ij} = 0 \quad (14)$$

using $\theta^{nl} \partial_n \partial_l = 0$. The Hamiltonian then takes the form of

$$\hat{H} = \frac{1}{2m} (\hat{g}^{ij} \hat{p}_i \hat{p}_j - i \partial_i \hat{g}^{ij} \hat{p}_j) \quad (15)$$

with $\hat{g}^{ij}(\hat{\mathbf{x}}, t) = \delta^{ij} + \hat{\gamma}^{ij}(\hat{\mathbf{x}}, t)$. We mention that the above Hamiltonian, despite presence of combinations of position and momentum operators, is free from ordering ambiguity. The above Hamiltonian in the position basis takes the form of

$$H_{\mathbf{x} \text{ basis}} = \frac{-1}{2m} (g^{ij} \partial_i \partial_j + \partial_i g^{ij} \partial_j) \quad (16)$$

According to recipe the Hamiltonian operator of a free particle in position basis is given by the Laplacian constructed by the metric $g^{ij}(\mathbf{x}, t)$,

$$H_{\mathbf{x} \text{ basis}} = -\frac{1}{2m} \nabla^2 = -\frac{1}{2m} \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} \partial_j) \quad (17)$$

in which $g = \det g_{ij}$. So interpreting (16) as the Hamiltonian of kind (17) gives the uni-modular condition $\det g_{ij} = 1$, which for acceptable order of θ says

$$\text{Tr } \gamma^{ij} = \gamma^i_i \propto \theta^{ij} F_{ij} = 0 \quad (18)$$

with $F_{ij} = \partial_i A_j - \partial_j A_i$. We mention that the condition above can not be considered as a gauge fixing one, because F_{ij} is gauge invariant, at least at this order of θ . So according to the construction above, for NC U(1) background $\mathbf{A}(\mathbf{x}, t)$ that satisfies (18) the dynamics is described by the motion of particle in presence of the effective metric

$$g^{ij} = \delta^{ij} + \theta^{kj} \partial_k A^i + \theta^{ki} \partial_k A^j, \quad (19)$$

satisfying the conditions

$$\partial_i \partial_j g^{ij} = 0, \quad \det g_{ij} = 1. \quad (20)$$

It is useful to compare the construction above with that of [3], in which the case of a massless scalar field $\widehat{\varphi}$ is considered in presence of the NC background \widehat{A}_μ . Since the action has only one term, one can use the the first order Seiberg-Witten map

$$\begin{aligned}\widehat{A}_\mu &= A_\mu - \frac{1}{2}\theta^{\alpha\beta}A_\alpha(\partial_\beta A_\mu + F_{\beta\mu}) \\ \widehat{\varphi} &= \varphi - \theta^{\alpha\beta}A_\alpha\partial_\beta\varphi\end{aligned}\tag{21}$$

that turns the action to the one for the scalar field φ in curved background given by the A_μ -dependent effective metric $G^{\mu\nu}$ [3], for which we have

$$\det G_{\mu\nu} - \det \eta_{\mu\nu} \propto \frac{D-3}{2} \theta^{\alpha\beta} F_{\alpha\beta} + O(\theta^2),\tag{22}$$

in $D+1$ space-time dimensions. We see that in 3+1 dimensions, the uni-modular condition is satisfied automatically.

3 Classical Dynamics In NC U(1) Background

In this section we derive the classical equation of motion of a particle with NC charge. The NC QED is given by the action ($\hbar = c = 1$)

$$S = \int d^D x dt \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i \bar{\psi} \gamma^\mu \left(\partial_\mu \psi - i [A_\mu, \psi]_\star \right) - m \bar{\psi} \psi \right),\tag{23}$$

$$\eta^{\mu\nu} = \text{diag}(+1, -1, \dots, -1), \quad \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 \eta^{\mu\nu},$$

in which $\bar{\psi} = \psi^\dagger \gamma^0$, and field strength is defined by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i [A_\mu, A_\nu]_\star.\tag{24}$$

The action above is invariant under the gauge transformation (4). Under the gauge transformation, the field strength transforms as

$$F_{\mu\nu} \rightarrow F'_{\mu\nu} = U \star F_{\mu\nu} \star U^*\tag{25}$$

We mention that the transformations of gauge fields as well as the field strength of NC U(1) theory, together with self-interaction terms in the pure gauge sector in action above, shows that it should be regarded as a non-Abelian gauge theory. Based on these facts, here we use the Wong's approach, originally adopted for particles with non-Abelian charge [10], to derive the classical equations of motion of charges in presence of NC U(1) background. This formalism is presented for charges in fundamental representation in [11]. Here we consider charges in adjoint representation, and give a presentation appropriate for emergent gravity interpretation of the result. At the first order of NC parameter the Lagrangian takes the form of

$$L = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + i \bar{\psi} \gamma^\mu \partial_\mu \psi + i \bar{\psi} \gamma^\mu \theta^{\alpha\beta} \partial_\alpha A_\mu \partial_\beta \psi - m \bar{\psi} \psi + O(\theta^2),\tag{26}$$

by which the equation of motion for ψ is obtained to be:

$$\mathbf{i} \gamma^0 \partial_0 \psi + \mathbf{i} \gamma^i \partial_i \psi + \mathbf{i} \gamma^\mu \theta^{\alpha\beta} \partial_\alpha A_\mu \partial_\beta \psi - m\psi = 0 \quad (27)$$

As previous section we assume 1) noncommutativity is just in spatial directions: $\theta^{0i} = 0$, 2) $A_0 = 0$. So, the above equation appears in the form:

$$\mathbf{i} \gamma^0 \partial_0 \psi + \mathbf{i} \gamma^i \partial_i \psi + \mathbf{i} \theta^{ij} \gamma^k \partial_i A_k \partial_j \psi - m\psi = 0 \quad (28)$$

Taking above as a Schrödinger equation we read the corresponding Hamiltonian as

$$\hat{H} = \alpha^i \hat{p}_i + \theta^{ij} \alpha^k \partial_i A_k \hat{p}_j + m\gamma^0 \quad (29)$$

in which $\alpha^k = \gamma^0 \gamma^k$. The Heisenberg equations of motion are derived for the operators as well:

$$\dot{\hat{x}}^k = \mathbf{i} [\hat{H}, \hat{x}^k] = \alpha^k + \theta^{ik} \alpha^j \partial_i A_j \quad (30)$$

$$\dot{\hat{p}}_k = \mathbf{i} [\hat{H}, \hat{p}_k] = -\theta^{ij} \alpha^l \partial_k \partial_i A_l \hat{p}_j \quad (31)$$

From the first equation we have $\alpha^k = \dot{\hat{x}}^k - \theta^{ik} \dot{\hat{x}}^j \partial_i A_j + O(\theta^2)$, which gives by second equation:

$$\dot{\hat{p}}_k = -\theta^{ij} \dot{\hat{x}}^l \partial_k \partial_i A_l \hat{p}_j + O(\theta^2). \quad (32)$$

One might take above as the equation of motion p_k in the curved background as

$$p_k = m \tilde{g}_{kl} \dot{x}^l, \quad \tilde{g}_{nl}(\mathbf{x}, t) = \delta_{nl} - \tilde{\gamma}_{nl}(\mathbf{x}, t) \quad (33)$$

in which we have gone to the classical regime by dropping the hats. By this interpretation one finds

$$\ddot{x}^l + \frac{1}{2} \delta^{lk} \left(\partial_i \tilde{\gamma}_{jk} + \partial_j \tilde{\gamma}_{ik} - \partial_k \tilde{\gamma}_{ij} \right) \dot{x}^i \dot{x}^j + \partial_0 \tilde{\gamma}^{il} \dot{x}_i + O(\tilde{\gamma}^2) = 0 \quad (34)$$

with $\tilde{\gamma}^{ij} = \theta^{kj} \partial_k A^i + \theta^{ki} \partial_k A^j$, as the same coming in metric (19). We mention that above result is simply the dynamics of a particle in presence of a time-dependent metric.

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